

# COHEN-MACAULAY $r$ -PARTITE GRAPHS WITH MINIMAL CLIQUE COVER

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**ABSTRACT.** In this note, we give some necessary conditions for an  $r$ -partite graph such that the edge ring of the graph is Cohen-Macaulay. It is proved that if  $G$  is an  $r$ -partite Cohen-Macaulay graph which is covered by some disjoint cliques of size  $r$ , then the clique cover is unique.

## 1. INTRODUCTION

Mainly, after using the notion of simplicial complexes and its algebraic interpretation by R. Stanley in 1970s to prove the upper bound conjecture for number of simplicial spheres [8], this notion has been one of the main streams of research in commutative algebra. In this stream, characterization and classification of Cohen-Macaulay simplicial complexes have been extensively studied in the last decades. It is known that Cohen-Macaulay property of a simplicial complex and its level graph are coincide. Therefore, to characterize all simplicial complexes which are Cohen-Macaulay, is enough to characterize all graphs with this property [8].

To examine special classes of graphs, Estrada and Villarreal in [3] found some necessary conditions for bipartite graphs to be Cohen-Macaulay. Finally, Herzog and Hibi in [4] presented a combinatorial characterization for bipartite graphs equivalent to Cohen-Macaulay property of these graphs. This purely combinatorial method can not be generalized for  $r$ -partite graphs in general. Because, as shown in Example 2.3, Cohen-Macaulay property may depend on characteristics of the base field. In other hand, it is shown in [11], the corresponding graph to a simplicial complex, such that has the same Cohen-Macaulayness property is covered by minimal possible number of cliques. In this paper, we consider

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$r$ -partite graphs with a minimal clique cover and find a necessary condition for Cohen-Macaulayness of these graphs. More precisely, we prove that in a Cohen-Macaulay  $r$ -partite graph with a minimal clique cover, there is a vertex of degree  $r - 1$  and the cover is unique.

## 2. PRELIMINARIES

A simple graph is an undirected graph that has no loop and multiple edge. A finite graph is denoted by  $G = (V(G), E(G))$ , where  $V(G)$  is the set of vertices and  $E(G)$  is the set of edges. Let  $|V(G)| = n$ . We use  $[n] = \{1, 2, \dots, n\}$  as vertices of  $G$ . The complementary graph of  $G$  is the graph  $\bar{G}$  on  $[n]$  whose edge set  $E(\bar{G})$  consists of those edges  $\{i, j\}$  which are not in  $E(G)$ . An independent set of vertices is a set of pairwise nonadjacent vertices. An  $r$ -partite graph is a graph that the set of its vertices can be partitioned into  $r$  disjoint subsets such that each set is independent. A subset  $A \subset [n]$  is a minimal vertex cover of  $G$  if (i) each edge of  $G$  is incident with at least one vertex in  $A$ , and (ii) there is no proper subset of  $A$  with property (i). It is easy to check that any minimal vertex cover of a graph is complement set of a maximal independent set of the graph. A graph  $G$  is called unmixed (well-covered) if any two minimal vertex covers of  $G$  have the same cardinality. A clique in a graph is a set of pairwise adjacent vertices, and by a  $r$ -clique we mean a clique of size  $r$ . An  $r$ -matching is a set of pairwise disjoint  $r$ -cliques and a perfect  $r$ -matching is an  $r$ -matching which covers all vertices of  $G$ .

Let  $\omega(G)$  denote the maximum size of cliques in  $G$ , which is called clique number of  $G$ . Let  $f : V(G) \rightarrow [k]$  be a map such that if  $v_1$  is adjacent to  $v_2$  then  $f(v_1) \neq f(v_2)$ . If such a map exists, we say that  $G$  is colorable by  $k$  colors. The smallest such  $k$  is called chromatic number of the graph and is denoted by  $\chi(G)$ . A graph  $G$  is called perfect if  $\omega(H) = \chi(H)$  for each induced subgraph  $H$  of  $G$ . The class of perfect graphs plays an important role in graph theory and most of computations in this class can be done by fast algorithms. L. Lovász in [7] has proved that a graph is perfect if and only if its complement is perfect. Chudnovsky et al in [2] have proved that a necessary and sufficient condition for a graph  $G$  to be perfect is that  $G$  does not have an odd hole (a cycle of odd length greater than 3) or an odd antihole (complement of an odd hole) as induced subgraph.

Let  $G$  be a graph on  $[n]$ . Let  $S = K[x_1, \dots, x_n]$ , the polynomial ring over a field  $K$ . The edge ideal  $I(G)$  of  $G$  is defined to be the ideal of  $S$  generated by all square-free monomials  $x_i x_j$  provided that  $i$  is adjacent

to  $j$  in  $G$ . The quotient ring  $R(G) = S/I(G)$  is called the edge ring of  $G$ .

Let  $R$  be a commutative ring with an identity. The depth of  $R$ , denoted by  $\text{depth}(R)$ , is the largest integer  $r$  such that there is a sequence  $f_1, \dots, f_r$  of elements of  $R$  such that  $f_i$  is not a zero-divisor in  $R/(f_1, \dots, f_{i-1})$  for all  $1 \leq i \leq r$ , and  $(f_1, \dots, f_r) \neq R$ . Such a sequence is called a regular sequence. The depth is an important invariant of a ring. It is bounded by another important invariant, the Krull dimension, the length of the longest chain of prime ideals in the ring. A ring  $R$  is called Cohen-Macaulay if  $\text{depth}(R) = \dim(R)$ . A graph  $G$  is called Cohen-Macaulay if the ring  $R(G)$  is Cohen-Macaulay.

**Theorem 2.1.** [9, Proposition 6.1.21] *If  $G$  is a Cohen-Macaulay graph, then  $G$  is unmixed.*

A simplicial complex  $\Delta$  on  $n$  vertices is a collection of subsets of  $[n]$  such that the following conditions hold:

- (i)  $\{i\} \in \Delta$  for each  $i \in [n]$ ,
- (ii) if  $E \in \Delta$  and  $F \subseteq E$  then  $F \in \Delta$ .

An element of  $\Delta$  is called a face and a maximal face with respect to inclusion is called a facet. The set of all facets of  $\Delta$  is denoted by  $\mathcal{F}(\Delta)$ . The dimension of a face  $F \in \Delta$  is defined to be  $|F| - 1$  and dimension of  $\Delta$  is maximum of dimension of its faces. A simplicial complex is called pure if all of its facets have the same dimension. For more details on simplicial complexes see [8].

The clique complex of a finite graph  $G$  on  $[n]$  is the simplicial complex  $\Delta(G)$  on  $[n]$  whose faces are the cliques of  $G$ . Let  $\Delta$  be a simplicial complex on  $[n]$ . We say that  $\Delta$  is shellable if its facets can be ordered as  $F_1, F_2, \dots, F_m$  such that for all  $j \geq 2$  the subcomplex  $(F_1, \dots, F_{j-1}) \cap F_j$  is pure of dimension  $\dim F_j - 1$ . An order of the facets satisfying this condition is called a shelling order. To say that  $F_1, F_2, \dots, F_m$  is a shelling order of  $\Delta$  is equivalent to say that for all  $i$ ,  $2 \leq i \leq m$  and all  $j < i$ , there exists  $l \in F_i \setminus F_j$  and  $k < i$  such that  $F_i \setminus F_k = \{l\}$ .  $G$  is called shellable if  $\Delta(\bar{G})$  has this property.

Let  $\Delta$  be a simplicial complex on  $[n]$  and  $I_\Delta$  be the ideal of  $S = K[x_1, \dots, x_n]$  generated by all square-free monomials  $x_{i_1} \cdots x_{i_t}$ , provided that  $\{i_1, \dots, i_t\}$  is not a face of  $\Delta$ . The ring  $S/I_\Delta$  is called the Stanley-Reisner ring of  $\Delta$ . A simplicial complex is called Cohen-Macaulay if its Stanley-Reisner ring is Cohen-Macaulay.

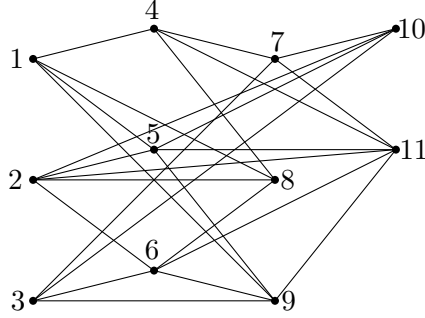


FIGURE 1. Cohen-Macaulay property depends on characteristic

**Theorem 2.2.** [5, Theorem 8.2.6] *If  $\Delta$  is a pure and shellable simplicial complex, then  $\Delta$  is Cohen-Macaulay.*

M. Estrada and R. H. Villarreal in [3] have proved that for a bipartite graph  $G$  Cohen-Macaulayness and pure shellability are equivalent. This is not true in general for  $r$ -partite graphs when  $r > 2$  (Example 2.3).

Also in bipartite graphs, Cohen-Macaulayness does not depend on characteristics of the ground field. But again, this is not true in general as shown in the following example.

**Example 2.3.** Let  $G$  be the graph in Figure 1. Then,  $R(G)$  is Cohen-Macaulay when the characteristic of the ground field  $K$  is zero but it is not Cohen-Macaulay in characteristic 2. Therefore the graph  $G$  is not shellable ([6]).

### 3. COHEN-MACAULAY PROPERTY AND UNIQUENESS OF PERFECT $r$ -MATCHING

M. Estrada and R. H. Villarreal in [3] have proved that if  $G$  is a Cohen-Macaulay bipartite graph and has at least one vertex of positive degree, then there is a vertex  $v$  such that  $\deg(v) = 1$ . By  $\deg(v)$  we mean the number of vertices adjacent to  $v$ . J. Herzog and T. Hibi in [4] have proved that a bipartite graph  $G$  with parts  $V_1$  and  $V_2$  is Cohen-Macaulay if and only if,  $|V_1| = |V_2|$  and there is an order on the vertices of  $V$  and  $W$  as  $v_1, \dots, v_n$  and  $w_1, \dots, w_n$  respectively, such that:

- 1)  $v_i \sim w_i$  for  $i = 1, \dots, n$ ,
- 2) if  $v_i \sim w_j$ , then  $i \leq j$ ,
- 3) for each  $1 \leq i < j < k \leq n$  if  $v_i \sim w_j$  and  $v_j \sim w_k$ , then  $v_i \sim w_k$ .

R. Zaare-Nahandi in [10] has proved that a well-covered bipartite graph  $G$  is Cohen-Macaulay if and only if there is a unique perfect 2-matching in  $G$ .

Let  $\alpha(G)$  denote the maximum cardinality of independent sets of vertices of  $G$ . Let  $\mathcal{G}$  be the class of graphs such that for each  $G \in \mathcal{G}$  there are  $k = \alpha(G)$  cliques in  $G$  covering all its vertices. For each  $G \in \mathcal{G}$  and cliques  $Q_1, \dots, Q_k$  such that  $V(Q_1) \cup \dots \cup V(Q_k) = V(G)$ , we may take  $Q'_1 = Q_1$  and for  $i = 2, \dots, k$ ,  $Q'_i$  the induced subgraph on the vertices  $V(Q_i) \setminus (V(Q_1) \cup \dots \cup V(Q_{i-1}))$ . Then  $Q'_1, \dots, Q'_k$  are  $k$  disjoint cliques covering all vertices of  $G$ . We call such a set of cliques, a basic clique cover of the graph  $G$ . Therefore any graph in the class  $\mathcal{G}$  has a basic clique cover.

**Proposition 3.1.** *Let  $G$  be an  $r$ -partite, unmixed and perfect graph such that all maximal cliques are of size  $r$ . Then  $G$  is in the class  $\mathcal{G}$ .*

*Proof.* Let  $V_1, \dots, V_r$  be parts of  $G$ . By [11],  $|V_1| = |V_2| = \dots = |V_r| = \alpha(G)$ . Also by [7], the complement graph  $\bar{G}$  is perfect. In other hand,  $V_i$  is a clique of maximal size in  $\bar{G}$  for each  $1 \leq i \leq r$ . Therefore,  $\chi(\bar{G}) = \omega(\bar{G}) = \alpha(G)$ . This implies that  $\bar{G}$  is  $\alpha(G)$ -partite. Therefore there are  $\alpha(G)$  disjoint maximal cliques in  $G$  covering all vertices.  $\square$

The converse of the above proposition is not true in the sense of the following example.

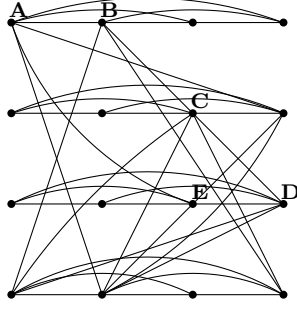
**Example 3.2.** Let  $G$  be the graph in Figure 2. Then  $G$  is a graph in class  $\mathcal{G}$  which is 4-partite, unmixed and all maximal cliques are of size 4. But the induced subgraph on  $\{A, B, C, D, E\}$  is a cycle of length 5 and therefore, by [2], the graph  $G$  is not perfect.

Let  $H$  be a graph and  $v$  be a vertex of  $H$ . Let  $N(v)$  be the set of all vertices of  $H$  adjacent to  $v$ .

**Theorem 3.3.** [9, Proposition 6.2.4] *If  $H$  is Cohen-Macaulay and  $v$  is a vertex of  $H$ , then  $H \setminus (v, N(v))$  is Cohen-Macaulay.*

**Theorem 3.4.** [11] *Let  $G$  be an  $r$ -partite unmixed graph such that all maximal cliques are of size  $r$ . Then all parts have the same cardinality and there is a perfect 2-matching between each two parts.*

Now, we present the main theorem of this paper which is generalization of [3, Theorem 2.4].

FIGURE 2. A graph in class  $\mathcal{G}$  which is not perfect

**Theorem 3.5.** *Let  $G$  be an  $r$ -partite graph in the class  $\mathcal{G}$  such that each maximal clique is of size  $r$ . If  $G$  is Cohen-Macaulay then there is a vertex of degree  $r - 1$  in  $G$ .*

*Proof.* By Theorem 3.4 all parts have the same cardinality. So there is a positive integer  $n$  such that  $|V| = rn$ . Assume that for all vertices  $v$  in  $G$  we have  $\deg(v) \geq r$ . Let  $Q_i = \{x_{1i}, x_{2i}, \dots, x_{ri}\}$  for  $i = 1, \dots, n$  are cliques in a basic clique cover of  $G$ . Without loss of generality, assume that  $v_{11}$  be a vertex of the minimal degree. If  $\deg(v_{11}) = (r - 1)n$  then  $G = K_{n,n,\dots,n}$  is a complete  $r$ -partite graph. Thus  $G$  is not Cohen-Macaulay by [1, Exercise 5.1.26] and we get a contradiction. Therefore,  $r \leq \deg(v_{11}) \leq (r - 1)n - 1$ .

Let  $N(v_{11}) = \{v_{21}, \dots, v_{2l_2}, v_{31}, \dots, v_{3l_3}, \dots, v_{r1}, \dots, v_{rl_r}\}$ . We have  $\deg(v_{11}) = l_2 + \dots + l_r$ . Without loss of generality, we may assume that  $l_2 \leq l_i$  for  $i = 3, \dots, r$ . Set  $G' = G \setminus (\{v_{11}\}, N(v_{11}))$ . The graph  $G'$  is Cohen-Macaulay by Theorem 2.1. If  $l_2 = 1$ , then, there exists  $3 \leq i \leq r$  such that  $l_i \geq 2$ . The sets

$$\{v_{12}, \dots, v_{1n}, v_{22}, \dots, v_{2n}, v_{3(l_3+1)}, \dots, v_{3n}, \dots, \widehat{(v_{i(l_i+1)}, \dots, v_{in})}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\}$$

and

$$\{v_{12}, \dots, v_{1n}, v_{3(l_3+1)}, \dots, v_{3n}, \dots, v_{i(l_i+1)}, \dots, v_{in}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\}$$

are two minimal vertex covers for  $G'$  and their cardinalities are not equal. Here, by  $\widehat{(v_{i(l_i+1)}, \dots, v_{in})}$  we mean the vertices  $v_{i(l_i+1)}, \dots, v_{in}$  are removed from the set. This contradicts to Cohen-Macaulayness of

$G'$ . Therefore,  $l_2 \geq 2$ . We claim that

$$\deg(v_{1i}) = l_2 + l_3 + \cdots + l_r = \deg(v_{11}), \quad i = 1, \dots, l_2.$$

It is enough to show that  $\deg(v_{12}) = l_2 + l_3 + \cdots + l_r$  and analogous argument proves the claim. If  $\deg(v_{12}) > l_2 + l_3 + \cdots + l_r$ , then there is a  $j_t$ ,  $l_t + 1 \leq j_t \leq n$  for some  $2 \leq t \leq r$ , such that  $v_{12} \sim v_{tj_t}$ . Without loss of generality we assume that  $t = 2$ .

If there is  $j_2$ ,  $l_2 + 1 \leq j_2 \leq n$ , such that  $v_{12} \sim v_{2j_2}$  then there is a minimal vertex cover for  $G'$  containing the set

$$\{v_{12}, v_{1(l_2+1)}, \dots, v_{1n}, v_{3(l_3+1)}, \dots, v_{3n}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\}.$$

In other hand,  $\{v_{2(l_2+1)}, \dots, v_{2n}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\}$  is a minimal vertex cover of  $G'$ . By  $l_2 \geq 2$  and Theorem 2.1, this contradicts Cohen-Macaulayness of  $G'$ . Therefore  $\deg(v_{12}) = l_2 + l_3 + \cdots + l_r$ . Thus, for all  $1 \leq i \leq l_2$  we have  $N(v_{1i}) = \{v_{21}, \dots, v_{2l_2}, v_{31}, \dots, v_{3l_3}, \dots, v_{r1}, \dots, v_{rl_r}\}$ . Consider the graph  $H = G \setminus (\{v_{2(l_2+1)}, \dots, v_{2n}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\} \cup N(v_{2(l_2+1)}) \cup \cdots \cup N(v_{2n}) \cup \cdots \cup N(v_{r(l_r+1)}) \cup \cdots \cup N(v_{rn}))$ . By Theorem 3.3,  $H$  is Cohen-Macaulay but the complement of  $H$  is not connected. This is a contradiction by [1, Exercise 5.1.26].  $\square$

Theorem 3.5 implies that the perfect  $r$ -matching in a Cohen-Macaulay  $r$ -partite graph is unique.

**Corollary 3.6.** *Let  $G$  be an  $r$ -partite graph in the class  $\mathcal{G}$  such that all maximal cliques are of size  $r$ . If  $G$  is Cohen-Macaulay then there is a unique perfect  $r$ -matching in  $G$ .*

*Proof.* Since  $G$  is in the class  $\mathcal{G}$ , there is a perfect  $r$ -matching in  $G$ . By Theorem 3.5, there is a vertex  $x \in V(G)$  of degree  $r - 1$ . Therefore, the  $r$ -clique in the  $r$ -matching which contains  $x$ , must be in all perfect  $r$ -matchings of  $G$ . The graph  $G \setminus (\{x, N(x)\})$  is again an  $r$ -partite graph in the class  $\mathcal{G}$  which is Cohen-Macaulay by Theorem 3.3. Continuing this process, we find that the chosen perfect  $r$ -matching is the unique perfect  $r$ -matching in  $G$ .  $\square$

## REFERENCES

- [1] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1998.
- [2] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem, *Ann. of Math.*, **164** (2006), no. 1, 51–229.
- [3] M. Estrada and R. H. Villarreal, Cohen-Macaulay bipartite graphs, *Arc. Math.*, **68** (1997), no. 2, 124–128.

- [4] J. Herzog and T. Hibi, Distributive lattices, bipartite graphs, and Alexander duality, *J. Algebraic Combin.*, **22** (2005), no. 3, 289–302.
- [5] J. Herzog and T. Hibi, *Monomial Ideals*, Springer-Verlag, 2011.
- [6] M. Katzman, Characteristic-independence of Betti numbers of graph ideals, *J. Combin. Theory, Series A* **113** (2006), no. 3, 435–454.
- [7] L. Lovász, A Characterization of Perfect Graphs, *J. Combin. Theory, Series B* **13** (1972), 95–98.
- [8] R. Stanley, *Combinatorics and Commutative Algebra*, 2nd Ed., Progress in Math., Birkhauser, 1996.
- [9] R. H. Villarreal, *Monomial Algebras*, Marcel Dekker, 2001.
- [10] R. Zaare-Nahandi, Cohen-Macaulayness of bi-partite graphs: revisited, Preprint. [arxiv:1012.0457v2](#) [[math.AC](#)]
- [11] R. Zaare-Nahandi, Pure simplicial complexes and well-covered graphs, Preprint. [arxiv: 1104.4556v2](#) [[math.AC](#)]

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